

# Thermodynamics of ideal Fermi gas under generic power law potential in $d$ -dimension

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Thermodynamics of ideal Fermi gas trapped in an external generic power law potential  $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$  are investigated systematically from the grand thermodynamic potential in  $d$  dimensional space. These properties are explored deeply in the degenerate limit ( $\mu \gg K_B T$ ), where the thermodynamic properties are greatly dominated by Pauli exclusion principle. Pressure and energy along with the isothermal compressibility is non zero at  $T = 0K$ , denoting trapped Fermi system is quite live even at absolute zero temperature. The nonzero value of compressibility denotes zero point pressure is not a constant but depends on volume.

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## 1. Introduction

The constrained role of external potential can change the characteristic of quantum gases[1, 2, 3, 4, 5, 6]. An increasing attraction towards this subject was noticed, after it was possible to create Bose-Einstein condensation (BEC) in magnetically trapped alkali gases[7, 8, 9]. A lot of studies are performed to understand the behaviour of ideal Bose gas[10, 11, 12, 13, 14] as well as ideal Fermi gas[11, 12]. But unlike the Bose gas, Fermi gas does not condensate due to the Pauli exclusion principle, the question of a large number of particles occupying a single energy state does not even arise in this case. At sufficiently low temperatures, Fermi gas displays its own brand of interesting behavior[11, 12], as its fugacity  $z_F$  can take on unrestricted values:  $0 < z_F < \infty$ [11] unlike the Bose gas who has restricted value of fugacity  $0 < z_B \leq 1$ [11]. The behavior of thermodynamic quantities of Fermi gas are remarkably governed by Pauli exclusion principle. For instance, the ground state pressure also known as degeneracy pressure[11] in ideal Fermi gas is nonzero unlike Bose gas or classical gas[11]. Nevertheless, a lot of efforts are made to understand different properties of Fermi system such as magnetism[15], conductivity[16], transport properties[17], equivalence with ideal Bose gas [18, 19], dimensionality effects[20], degeneracy[21], polylogarithms[22], q-deformed system[23, 24].

In a real systems, of course interaction between particles does exist. But taking it into account makes the problem difficult to solve analytically. Nevertheless, to understand the effect of interactions in quantum gases and to retain the essential physics, we approximately represent the real system by non interacting particles in the presence of an external potential[2, 25, 26]. The trapping potential in atomic gases provide the opportunity to manipulate the quantum statistical effects. Some drastic changes are noted in case of Bose system[1, 2] under trapping potential. For instance BEC is possible in  $d = 2$  in trapped Bose gas which was not a case in ideal Bose gas[1, 2]. Therefore, it will be interesting to investigate how the trapping potential do change the properties of Fermi gas. Li et al.[?], in their work has presented internal energy, heat capacity, ground state energy of Fermi gas under spherically symmetric potential ( $U = br^t$ ) in arbitrary dimension. However, in the present study we have investigated properties of ideal Fermi gas under generic power law potential  $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$  in  $d$ -dimension, which will be symmetric under certain choice  $n_i$ ,  $x_i$  and  $c_i$ . So, we can reconstruct the results of Li et al. [?] choosing this special condition. At first, we have calculated the density of states which enables us to determine the grand potential. Then from the grand potential of

the system, we have derived the thermodynamic quantities such as internal energy  $E$  entropy  $S$ , pressure  $P$ , number of particle  $N$ , helmholtz free energy  $A$ , isothermal compressibility  $\kappa_T$ , specific heat at constant volume  $C_V$  and pressure  $C_P$  and their ratio. In the high temperature limit, the thermodynamic quantities of free quantum gases reduce to the form of classical gas[11]. Same trend is observed in case of trapped system[1] too. Therefore, the low temperature limit of the thermodynamic quantities of quantum gases are particularly important, as the true quantum nature is explicit in that region. In Bose system, low temperature limit refers to condensed phase. It was found trapping potential changes the general criterion of BEC as well as the condition of jump of specific heat[1, 2]. So, low temperature limit of thermodynamic quantities related to trapped Fermi system have been investigated using Sommerfield expansion[11]. It will be very intriguing to investigate energy and pressure of trapped Fermi gas while the gas is in the degenerate limit to check whether they remain nonzero under generic trapping potential at  $T = 0K$ . Isothermal compressibility (inverse of bulk modulus) is also calculated to check whether the ground state pressure has volume dependency or it is merely just a constant. A point to note that, in the hamiltonian instead of  $\frac{p^2}{2m}$  type kinetic part, we have took  $ap^s$ , where  $p$  is momentum,  $a$  is constant and  $s$  is a arbitrary kinematic parameter. Different kinematic characteristics of quantum systems lead to different characteristics[1, 28, 29]. So, choosing arbitrary kinematic parameter, we make the significant conclusions in more generalized way. In the current study it is found the concept of effective volume plays an important role in trapped Fermi gas, as seen in trapped Bose gas[1].

The report is organized in the following way. The density of states and grand potential are calculated in section 2. Section 3 is devoted to investigate the thermodynamic quantities. Properties of degenerate Fermi gas is presented in section 4. Results and discussions are presented in section 5. The report is concluded in section 6.

## 2. Density of States and grand potential of Fermi gas under generic power law potential in d dimension

Considering the ideal Fermi gas with kinematic parameter  $l$  in a confining external potential in a  $d$ -dimensional space with energy spectrum,

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i} \quad (1)$$

Where,  $b, l, a_i, c_i, n_i$  are all positive constants,  $p$  is the momentum and  $x_i$  is the  $i$  th component of coordinate of a particle. Here,  $c_i, a_i$  and  $n_i$  determines the depth and confinement power of the potential. With  $l = 2$ ,  $b = \frac{1}{2m}$  one can get the energy spectrum of the hamiltonian used in the literatures [11, 12, 10, 2]. For the free system all  $n_i \rightarrow \infty$ .

Density of states can be obtained from the following formula,

$$\rho(\epsilon) = \int \int \frac{d^d r d^d p}{(2\pi\hbar)^d} \delta(\epsilon - \epsilon(p, r)) \quad (2)$$

So, from the above equation density of states is,

$$\rho(\epsilon) = B \frac{\Gamma(\frac{d}{l} + 1)}{\Gamma(\chi)} \epsilon^{\chi-1} \quad (3)$$

where,

$$B = \frac{gV_d C_d}{h^d a^{d/l}} \prod_{i=1}^d \frac{\Gamma(\frac{1}{n_i} + 1)}{c_i^{\frac{1}{n_i}}} \quad (4)$$

Here,  $C_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ ,  $g$  is the spin degeneracy factor,  $V_d = 2^d \prod_{i=1}^d a_i$  is the volume of an  $d$ -dimensional rectangular whose  $i$ -th side has length  $2a_i$ .  $\Gamma(l) = \int_0^\infty dx x^{l-1} e^{-x}$  is the gamma function and  $\chi = \frac{d}{l} + \sum_{i=1}^d \frac{1}{n_i}$ .

The grand potential for the Fermi system,

$$q = - \sum_{\epsilon} \ln(1 + z \exp(-\beta \epsilon)) \quad (5)$$

$\beta = \frac{1}{kT}$ , where  $k$  being the Boltzmann Constant and  $z = \exp(\beta \mu)$  is the fugacity, where  $\mu$  being the chemical potential. Using the Thomas-Fermi semiclassical approximation[25] and re-writing the previous equation,

$$q = q_0 - \int_0^\infty \rho(\epsilon) \ln(1 + z \exp(-\beta \epsilon)) \quad (6)$$

So, using the density of states of Eq. (3) we finally get the grand potential,

$$q = q_0 + B \Gamma\left(\frac{d}{l} + 1\right) (kT)^\chi f_{\chi+1}(z) \quad (7)$$

where,  $q_0 = -\ln(1 - z)$  and  $f_l(z)$  is the Fermi function which is defined as,

$$f_p(z) = \int_0^\infty dx \frac{x^{p-1}}{z^{-1}e^x + 1} = \sum_{j=1}^\infty (-1)^{j-1} \frac{z^j}{j^p} \quad (8)$$

### 3. Thermodynamics of Fermi gas under generic power law potential in d dimension

#### 3.1. Number of Particle

The number of particles  $N$  can be obtained,

$$N = z \left( \frac{\partial q}{\partial z} \right)_{\beta, V} = N_0 + \frac{g C_n \Gamma\left(\frac{d}{l} + 1\right) V_d \prod_{i=1}^d \Gamma\left(\frac{1}{n_i} + 1\right)}{h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i}} (kT)^\chi f_\chi(z) \quad (9)$$

Here,  $N_0 = \frac{z}{1-z}$  is the ground state occupation number.

Now, defining,

$$V'_d = V_d \prod_{i=1}^d \left( \frac{kT}{c_i} \right)^{1/n_i} \Gamma\left(\frac{1}{n_i} + 1\right) \quad (10)$$

$$\lambda' = \frac{h b^{\frac{1}{l}}}{\pi^{\frac{1}{2}} (kT)^{\frac{1}{l}}} \left[ \frac{d/2 + 1}{d/l + 1} \right]^{1/d} \quad (11)$$

It is noteworthy,

$$\lim_{n_i \rightarrow \infty} V'_d = V_d \quad (12)$$

$$\lim_{n_i \rightarrow \infty} \chi = \frac{d}{l} \quad (13)$$

$$\lim_{l \rightarrow 2, b \rightarrow \frac{1}{2m}} \lambda' = \lambda_0 = \frac{h}{(2\pi m kT)^{1/2}} \quad (14)$$

So, if we choose  $l = 2$  and  $b = \frac{1}{2m}$  from Eq. (14) we get  $\lambda_0 = \frac{h}{(2\pi mkT)^{1/2}}$ , which is the thermal wavelength of nonrelativistic free massive fermions. However, it should be noted that, when  $l \neq 2$ ,  $\lambda'$  then depends on dimension. With  $d = 3$  and  $d = 2$ , thermal wavelength of photons are respectively  $\frac{hc}{2\pi^{1/2}kT}$  and  $\frac{hc}{(2\pi)^{1/2}kT}$  which can be obtained from from Eq. (11) by choosing  $b = c$ , where  $c$  being the velocity of light. So, one can reproduce the thermal wavelength of both massive and massless fermions from the definition of  $\lambda'$  with more general energy spectrum. But one needs to consider the effects of antiparticles to calculate the thermodynamic quantities of ultrarelativistic quantum gas[27].

The number of particle equation is then written as,

$$N - N_0 = g \frac{V_d'}{\lambda'^d} f_\chi(z) \quad (15)$$

The number of particle equation for free massive fermions (with  $l = 2$ ,  $a = \frac{1}{2m}$ , all  $n_i \rightarrow \infty$ ) in  $d$  dimensional space can be obtained from Eq. (15),

$$N - N_0 = g \frac{V_d}{\lambda_0^d} f_{\frac{d}{2}}(z) \quad (16)$$

which gives the exact equation for number of particles at  $d = 3$ [11, 12].

### 3.2. Internal Energy

From the Grand Canonical Ensemble internal energy  $E$  is,

$$\begin{aligned} E &= -\left(\frac{\partial q}{\partial \beta}\right)_{z,V} \\ &= \frac{g C_n \Gamma(\frac{d}{l} + 1) V_d \prod_{i=1}^d \Gamma(\frac{1}{n_i} + 1)}{h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i}} (kT)^{\chi+1} f_{\chi+1}(z) \end{aligned} \quad (17)$$

$$= NkT \chi \frac{f_{\chi+1}(z)}{f_\chi(z)} \quad (18)$$

In case of free massive fermions,

$$E = NkT \frac{d}{2} \frac{f_{d/2+1}(z)}{f_{d/2}(z)} \quad (19)$$

which is in accordance with the exact expression of  $E$  for  $d = 3$ [11, 12].

Now as  $T \rightarrow \infty$ , from Eq. (19) it is seen, the internal energy becomes,  $E = NkT\chi$ . For free massive fermions it is  $E = \frac{d}{2}NkT$ , which becomes  $\frac{3}{2}NkT$ , when  $d = 3$ . Thus  $E$  approaches the classical value at high temperature. The exact same trend is also seen in case of Bose gas[1].

### 3.3. Entropy

The entropy  $S$  can be obtained from Grand Canonical Ensemble,

$$\begin{aligned} S &= kT \left(\frac{\partial q}{\partial T}\right)_{z,V} - Nk \ln z + kq \\ &= Nk \left[ \frac{v_d'}{\lambda'^d} (\chi + 1) f_{\chi+1}(z) - \ln z \right] \end{aligned} \quad (20)$$

As before, for free massive fermions, one can find Eq. (20) approaches to,

$$S = Nk[\frac{v_d}{\lambda^d}(\frac{d}{2} + 1)f_{\frac{d}{2}+1}(z) - \ln z] \quad (21)$$

Again at  $d = 3$ , Eq. (21) reduces to same expression for entropy as Ref.[11, 12]

### 3.4. Helmholtz Free Energy

From the Grand Canonical Ensemble we get the expression of Helmholtz Free Energy for Fermi system,

$$\begin{aligned} A &= -kTq + NkT \ln z \\ &= -NkT \frac{f_{\chi+1}(z)}{f_{\chi}(z)} + NkT \ln z \end{aligned} \quad (22)$$

In case of free massive fermions, the above expression reduces like below,

$$\frac{A}{NkT} = -\frac{f_{\frac{d}{2}+1}(z)}{f_{\frac{d}{2}}(z)} + \ln z \quad (23)$$

Now, for  $d = 3$ , the above equation produces the exact expression for Helmholtz free Energy[11, 12].

### 3.5. Pressure

Rewriting equation (15) stating the number of particles,

$$\frac{N - N_0}{V_d \prod_{i=1}^d (\frac{kT}{c_i})^{1/n_i} \Gamma(\frac{1}{n_i} + 1)} = \frac{N - N_0}{V'_d} = \frac{g}{\lambda'^d} g_{\chi}(z)$$

Now a very well known expression for the nonrelativistic  $d$ -dimensional ideal free Free gas[11],

$$\frac{N - N_0}{V_d} = \frac{g}{\lambda_0^d} f_{d/2}(z)$$

Comparing the above equations, we can say  $V'_d$  is a more generalized extension of  $V_d$ . Where,

$$V'_d = V_d \prod_{i=1}^d (\frac{kT}{c_i})^{1/n_i} \Gamma(\frac{1}{n_i} + 1)$$

It represents the effect of external potential on the performance of trapped fermions. Calling  $V'_d$  the effective volume the grand potential can be rewritten as,

$$q = q_0 + g \frac{g V'_d}{\lambda'^d} f_{\chi+1}(z) \quad (24)$$

So, the effective pressure

$$P' = \frac{1}{\beta} (\frac{\partial q}{\partial V'_d}) = \frac{gkT}{\lambda'^d} f_{\chi+1}(z) \quad (25)$$

Which can be rewritten as,

$$P' = \frac{NkT}{V_d'} \frac{f_{\chi+1}(z)}{f_{\chi}(z)} \quad (26)$$

The above equation is very general equation of state for any dimensionality  $d$ , any dispersion relation of the form ( $\propto p^s$ ) having any form of generic power law trap and obviously it is expected that it will reproduce the special case of free system. For free system the equation (26) becomes,

$$P = \frac{1}{\beta} \left( \frac{\partial q}{\partial V_d} \right) = \frac{NkT}{V_d} \frac{f_{d/2+1}(z)}{f_{\frac{d}{2}}(z)} \quad (27)$$

which is in accordance with Ref.[11, 12] at  $d=3$ .

Now, comparing Eq. (18) and (26) we get,

$$P'V_d' = \frac{E}{\chi} \quad (28)$$

For  $d$ -dimensional free Fermi gas one can obtain from previous equation

$$PV_d = \frac{2}{d} E \quad (29)$$

This is an important and familiar relation,  $PV = \frac{2}{3}E$  when  $d = 3$ [11, 10, 12, 13]. This actually shows equation (28) is a very significant relation for the Fermi gas irrespective whether they are trapped or free. And in case of trapped fermions effective volume and effective pressure plays the same role as volume and pressure in current textbooks and literatures. Interestingly, the Bose gas also maintains this equation.[1]

### 3.6. Heat Capacity

Heat capacity at constant volume  $C_v$ ,

$$\begin{aligned} C_V &= T \left( \frac{\partial S}{\partial T} \right)_{N,V} \\ &= Nk \left[ \chi(\chi+1) \frac{\nu'}{\lambda^{1/D}} f_{\chi+1}(z) - \chi^2 \frac{f_{\chi}(z)}{f_{\chi-1}(z)} \right] \end{aligned} \quad (30)$$

For Free massive fermions, the expression becomes,

$$C_V = Nk \left[ \frac{d}{2} \left( \frac{d}{2} + 1 \right) \frac{\nu}{\lambda^{1/D}} f_{\frac{d}{2}+1}(z) - \frac{d^2}{2} \frac{f_{\frac{d}{2}}(z)}{f_{\frac{d}{2}-1}(z)} \right] \quad (31)$$

And in the high temperature limit of  $C_v$  approaches its classical value as it becomes  $\chi Nk$  for trapped system and  $\frac{d}{2} Nk$  for free system, which is  $\frac{3}{2} Nk$ , when  $d = 3$ .

Now, heat capacity at constant pressure  $C_p$ ,

$$\begin{aligned} C_P &= T \left( \frac{\partial S}{\partial T} \right)_{N,P} \\ &= Nk \left[ (\chi+1)^2 f_{\chi+1}^2(z) f_{\chi-1}(z) \left( \frac{\nu'}{\lambda^{1/D}} \right)^3 - \chi(\chi+1) f_{\chi+1}(z) \frac{\nu'}{\lambda^{1/D}} \right] \end{aligned} \quad (32)$$

In case of free massive Fermi gas, the above equation reduces to,

$$C_P = Nk[(\frac{d}{2} + 1)^2 f_{\frac{d}{2}+1}^2(z) f_{\frac{d}{2}-1}(z) (\frac{\nu'}{\lambda^D})^3 - \frac{d}{2}(\frac{d}{2} + 1) f_{\frac{d}{2}+1}(z) \frac{\nu'}{\lambda^D}] \quad (33)$$

It coincides exactly with Ref. [11, 13] for  $d = 3$ . Again in the high temperature limit  $C_p$  becomes  $(\chi + 1)Nk$  for trapped system and  $(\frac{d}{2} + 1)Nk$  for free system, which is  $\frac{5}{2}Nk$ , when  $d = 3$ . So, in the high temperature limit  $C_p$  approaches its classical value.

Now, the ratio,  $\gamma = (\frac{C_P}{C_V})$  for Fermi gas is given by,

$$\gamma = \frac{(\chi + 1)^2 \frac{f_{\chi+1}^2(z) f_{\chi-1}(z)}{f_{\chi}^3(z)} - \chi(\chi + 1) \frac{f_{\chi+1}(z)}{f_{\chi}(z)}}{\chi(\chi + 1) \frac{\nu'}{\lambda^D} f_{\chi+1}(z) - \chi^2 \frac{f_{\chi}(z)}{f_{\chi-1}(z)}} \quad (34)$$

Now, in high temperature limit, the above equation becomes,

$$\gamma = \frac{(\chi + 1)^2 - \chi(\chi + 1)}{\chi(\chi + 1) - \chi^2} = 1 + \frac{1}{\chi} \quad (35)$$

In case of free system, choosing all  $n_i \rightarrow \infty$ , we get from the above equation,

$$\gamma = 1 + \frac{l}{d} \quad (36)$$

With  $d = 3$  and  $l = 2$ ,  $\gamma$  equals  $\frac{5}{3}$ , thus obtaining the classical value at high temperature limit.

### 3.7. Isothermal Compressibility

The Isothermal compressibility of Fermi gas can be obtained,

$$\begin{aligned} \kappa_T &= -V_d' \left( \frac{\partial V'}{\partial P'} \right)_{N,T} \\ &= -V_d' \left( \frac{\partial P'}{\partial z} \right)_{N,T} \left( \frac{\partial z}{\partial V'} \right)_{N,T} \\ &= \frac{V_d'}{NkT} \frac{f_{\chi-1}(z)}{f_{\chi}(z)} \end{aligned} \quad (37)$$

which reproduces the same result for isothermal compressibility of free massive Fermi gas at  $d = 3$  [11]. And as  $T \rightarrow \infty$   $\kappa_T$  takes the classical value for free system, which is  $\frac{1}{P}$ .

## 4. The thermodynamic properties of a degenerate Fermi gas under generic power law potential

At low temperature, can approximate the Fermi function and write it as quickly convergent Sommerfield series[11]

$$f_p(z) = \frac{(\ln z)^p}{\Gamma(p+1)} [1 + p(p-1) \frac{\pi^2}{6} \frac{1}{(\ln z)^2} + p(p-1)(p-2)(p-3) \frac{7\pi^4}{360} \frac{1}{(\ln z)^4} + \dots] \quad (38)$$

At  $T = 0K$ , taking only the first the first term in Eq (38). Substituting this into Eq. (15) we get

$$N - N_0 = N_e = \frac{gC_n\Gamma(\frac{d}{l} + 1)V_d \prod_{i=1}^d \Gamma(\frac{1}{n_i} + 1)}{h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i} \Gamma(\chi + 1)} E_F^\chi \quad (39)$$

Which turns out,

$$E_F = \left[ \frac{h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i} \Gamma(\chi + 1) N_e}{gC_n\Gamma(\frac{d}{l} + 1)V_d \prod_{i=1}^d \Gamma(\frac{1}{n_i} + 1)} \right]^{\frac{1}{\chi}} \quad (40)$$

Following the method of Ref.[4, 11, 12] we approximate the chemical potential and fugacity from Eq.(15),

$$\mu = kT \ln z = E_F \left[ 1 - (\chi - 1) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right] \quad (41)$$

Using these approximation we can calculate the thermodynamic quantities of the previous section,

$$\frac{E}{N} = \frac{\chi}{\chi + 1} E_F \left[ 1 + (\chi + 1) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right] \quad (42)$$

$$\frac{S}{Nk} = \frac{\chi \pi^2}{3E_F} kT \quad (43)$$

$$P = \frac{E_F N}{(\chi + 1)V'} \left[ 1 + (\chi + 1) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right] \quad (44)$$

$$\frac{C_V}{Nk} = \frac{\chi \pi^2}{3E_F} kT \quad (45)$$

$$\kappa_T = \frac{V' \chi}{NE_F} \left\{ 1 + (1 - \chi) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right\} \quad (46)$$

In case of free massive fermions (choosing  $l = 2$ ), Eq. (42)-(46) becomes,

$$\frac{E}{N} = \frac{d}{d + 2} E_F \left[ 1 + \left( \frac{d}{2} + 1 \right) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right] \quad (47)$$

$$\frac{S}{Nk} = \frac{d \pi^2}{6E_F} kT \quad (48)$$

$$P = \frac{2E_F N}{(d + 2)V'} \left[ 1 + \left( \frac{d}{2} + 1 \right) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right] \quad (49)$$

$$\frac{C_V}{Nk} = \frac{d \pi^2}{6E_F} kT \quad (50)$$

$$\kappa_T = \frac{V' d}{2NE_F} \left\{ 1 + \left( 1 - \frac{d}{2} \right) \frac{\pi^2}{6} \left( \frac{kT}{E_F} \right)^2 \right\} \quad (51)$$

At temperature  $T = 0K$ , entropy  $S = 0$  which is accordance will 3rd law of Thermodynamics. The internal energy, pressure and isothermal compressibility  $T = 0K$ ,

$$E_0 = \frac{\chi}{\chi + 1} NE_F \quad (52)$$

$$P_0 = \frac{1}{(\chi + 1)} \frac{N}{V'} E_F \quad (53)$$

$$\kappa_{T0} = \frac{V \chi}{NE_F} \quad (54)$$



In case of Free massive Fermi gas the above equations reduces to,

$$E_0 = \frac{d}{d+2} N E_F \quad (55)$$

$$P_0 = \frac{2}{(d+2)} \frac{N}{V} E_F \quad (56)$$

$$\kappa_{T0} = \frac{V}{N E_F} \frac{d}{2} \quad (57)$$

At  $d = 3$ , Eq. (55) and (56) become exactly same as in Ref. [11]

## 5. Discussion

Thermodynamics of ideal Fermi gas in the presence of an external generic power law potential are discussed in this section. It is seen the effective volume  $V'_d$  is a very salient feature of trapped system, playing the same role in trapped system as the volume in free system, which enables us to treat trapped fermi gas as well the bose gas[1] to be treated as a free one. Difference between  $V'_d$  and  $V_d$  is that, the former depends on temperature and power law exponent while the latter does not. But as all  $n_i \rightarrow \infty$ ,  $V'_d$  approaches  $V_d$ . In this process the more general thermal wavelength  $\lambda'$  is defined with arbitrary kinematic parameter in any dimension. It was shown how  $\lambda'$  can reproduce the thermal wavelengths of literatures in different dimensions. In case of trapped Fermi gases,  $V'_d$  and  $\lambda'$  enable all the thermodynamic functions of the system to be expressed in a compact form similar to those of free Fermi gas.

At first the density of states and grand potential is calculated in section 2. All the thermodynamic quantities are derived from the grand potential in section 3. It is seen that all the thermodynamic quantities for trapped Fermi gas are some function of Fermi functions, just as in the case of free Fermi gas. But in the former case the Fermi functions depend on  $\chi = \frac{d}{l} + \sum_{i=1}^d \frac{1}{n_i}$  and  $z$ , whether in the later case Fermi functions depend on  $\frac{d}{l}$  and  $z$ . And as  $n_i \rightarrow \infty$ , the mathematical form of thermodynamic quantities of trapped system reduce to that of free system. Nevertheless it is noteworthy that Eq. (28) is a very remarkable relation for quantum gases as both Bose[1] and Fermi system maintains it.

In general, the thermodynamic quantities of trapped system differ from free system. We can specifically check this by comparing free system with harmonically trapped potential. Let,  $d = 3$ ,  $a = \frac{1}{2m}$ ,  $l = 2$ ,  $n_1 = n_2 = n_3 = 2$ ,  $c_i = \frac{m\omega^2}{2}$  and  $g = 2$ . Results of some of the physical quantities have been listed in Table 1. From the table it is seen, thermodynamic quantities are affected by the trapping potential. The signature of trapping potential is present in the low as well as in the high temperature limit of thermodynamic functions. As we know chemical potential approaches Fermi Energy as  $T \rightarrow 0$  in case of free Fermi gas, same phenomena is also observed in case of trapped Fermi gas, although Fermi Energy do vary comparing trapped system with the free one. Let us turn our attention to low temperature limit of Fermi gas. It is seen both  $C_V$  and  $S$  has same numerical value in this limit just like the free system and goes to zero at  $T = 0K$ . The later actually is a manifestation third law of thermodynamics. But most significantly internal energy and pressure of trapped Fermi gas do not go to zero as  $T = 0K$  just as free Fermi gas. According to equation (46), no matter what power law exponent one chooses, ground state pressure never goes to zero. It suggests, the ground state energy and ground state pressure seen here is clearly a quantum effect arising due to Pauli exclusion principle due to which the system can not settle down into a single energy state as in the case of Bose gas. And therefore spread over a lowest available energy states. More interestingly, isothermal compressibility of Fermi system is nonzero at  $T = 0K$ , indicating zero point pressure is not merely a constant but depends on volume.

Table 1. Dissimilarity between free and harmonic-potential-trapped nonrelativistic Fermi gases in  $d = 3$ .

Physical quantity	Free gas	Trapped gas
Fermi Energy	$\frac{\hbar^2}{2m} (6\pi^2 (\frac{N}{V})^{\frac{2}{3}})$	$\hbar\omega (3N)^{1/3}$
Fermi Temperature	$\frac{\hbar^2}{2mk} (6\pi^2 (\frac{N}{V})^{\frac{2}{3}})$	$\frac{\hbar\omega}{k} (3N)^{1/3}$
Internal energy	$\frac{3}{2} NkT \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$	$3NkT \frac{f_4(z)}{f_3(z)}$
Internal energy at lower temperature	$\frac{3}{5} NE_F (1 + \frac{5}{2} \frac{\pi^2}{6} (\frac{kT}{E_F})^2)$	$\frac{3}{4} NE_F (1 + \frac{2\pi^2}{3} (\frac{kT}{E_F})^2)$
Ground state energy	$\frac{3}{5} NE_F$	$\frac{3}{4} NE_F$
Ground state pressure	$\frac{2}{5} \frac{N}{V} E_F$	$\frac{1}{4} \frac{N}{V} E_F$
Particle number at ground state	$\frac{4\pi V}{3h^3} (2mE_F)^{3/2}$	$\frac{1}{3} (\frac{E_F}{\hbar\omega})^3$
Internal energy at higher temperature	$\frac{3}{2} NkT$	$3NkT$
Isothermal Compressibility	$\frac{V}{NkT} \frac{f_{1/2}(z)}{f_{3/2}(z)}$	$\frac{V'}{NkT} \frac{f_2(z)}{f_3(z)}$
Isothermal Compressibility at higher temperature	$\frac{1}{P}$	$\frac{1}{P'}$
Isothermal Compressibility at lower temperature	$\frac{3V}{2NE_F} [1 - \frac{\pi^2}{12} (\frac{kT}{E_F})^2]$	$\frac{3V}{NE_F} [1 - \frac{\pi^2}{3} (\frac{kT}{E_F})^2]$
Isothermal Compressibility at $T = 0K$	$\frac{3}{2} \frac{V}{NE_F}$	$\frac{3V}{NE_F}$
Chemical potential at lower temperature	$E_F (1 - \frac{\pi^2}{12} (\frac{kT}{E_F})^2)$	$E_F (1 - \frac{\pi^2}{3} (\frac{kT}{E_F})^2)$
Chemical potential at $T = 0K$	$E_F$	$E_F$
Chemical potential at higher temperature	$-kT \ln[\frac{3}{4} \sqrt{\pi} (\frac{kT}{E_F})^{3/2}]$	$-kT \ln[6 (\frac{kT}{E_F})^3]$
$\frac{C_V}{Nk}$	$(\frac{15}{4}) \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - (\frac{9}{4}) \frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}$	$12 \frac{f_4(z)}{f_3(z)} - 9 \frac{f_3(z)}{f_2(z)}$
$\frac{C_V}{Nk}$ at lower temperature	$\frac{\pi^2}{2} \frac{kT}{E_F}$	$\pi^2 \frac{kT}{E_F}$
$\frac{C_V}{Nk}$ at higher temperature	$\frac{3}{2}$	3

## 6. Conclusion

From the grand potential, the thermodynamic properties of Fermi gas trapped under generic power law potential have been evaluated. The calculated physical quantities reduce to expressions available in the literature, with appropriate choice of power law exponents and dimensionality. The thermodynamic quantities are studied further closely in the degeneracy limit. It is found pressure, energy and isothermal compressibility is nonzero, with any trapping potential, indicating the governing power of Pauli exclusion principle. In this manuscript, we have restricted our discussion in case of ideal system under trapping potential. It will be very interesting to see the effect of interaction in the degeneracy limit.

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